

# Regularization of Mickelsson generators for non-exceptional quantum groups.

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## Abstract

Let  $\mathfrak{g}' \subset \mathfrak{g}$  be the pair of Lie algebras of either symplectic or orthogonal infinitesimal endomorphisms of the complex vector spaces  $\mathbb{C}^{N-2} \subset \mathbb{C}^N$  and  $U_q(\mathfrak{g}') \subset U_q(\mathfrak{g})$  the pair of quantum groups with triangular decomposition  $U_q(\mathfrak{g}) = U_q(\mathfrak{g}_-)U_q(\mathfrak{g}_+)U_q(\mathfrak{h})$ . Let  $Z_q(\mathfrak{g}, \mathfrak{g}')$  be the corresponding step algebra and regard its generators as rational trigonometric functions  $\mathfrak{h}^* \rightarrow U_q(\mathfrak{g}_\pm)$ . We describe their regularization such that the resulting generators do not vanish when specialized at any weight.

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## 1 Introduction

This rather technical paper is devoted to regularization of generators of Mickelsson algebras, regarded as meromorphic functions on the weight space. For a general theory of Mickelsson algebras, the reader is referred to [1, 2, 3] (the classical universal enveloping algebras) and [4, 5] (quantum groups). Here we are concerned with the special case related to the pair  $\mathfrak{g}' \subset \mathfrak{g}$  of Lie algebras of orthogonal/symplectic infinitesimal transformations of a fixed pair of vector spaces  $\mathbb{C}^{N-2} \subset \mathbb{C}^N$ .

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Let  $\mathfrak{g}' = \mathfrak{g}'_- \oplus \mathfrak{h}' \oplus \mathfrak{g}'_+$  be the triangular decomposition compatible with a decomposition  $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ , i.e.  $\mathfrak{g}'_{\pm} \subset \mathfrak{g}_{\pm}$ , with  $\mathfrak{h}' \subset \mathfrak{h}$  being the Cartan subalgebras. Let  $N(\mathfrak{g}, \mathfrak{g}')$  denote the normalizer of the left ideal  $U_q(\mathfrak{g})\mathfrak{g}'_+$ , i.e. the maximal subalgebra in  $U_q(\mathfrak{g})$  where  $U_q(\mathfrak{g})\mathfrak{g}'_+$  is a two-sided ideal. Then the quotient  $N(\mathfrak{g}, \mathfrak{g}')/U_q(\mathfrak{g})\mathfrak{g}'_+$  is called step or Mickelsson algebra and denoted by  $Z_q(\mathfrak{g}, \mathfrak{g}')$ . Its significance comes from the fact that it preserves the subspace of  $\mathfrak{g}'$ -singular vectors in  $\mathfrak{g}$ -modules (recall that a vector is called singular if it generates the trivial representation of  $U_q(\mathfrak{g}')$ ).

The Mickelsson algebra carries a two-sided  $U_q(\mathfrak{h})$ -action, and is generated by elements  $z_0, z_{\pm\alpha}$  of weights 0 and, respectively,  $\pm\alpha$  with  $\alpha \in R_{\mathfrak{g}}^+ - R_{\mathfrak{g}'}^+$  (the set of positive roots of  $\mathfrak{g}$  minus those of  $\mathfrak{g}'$ ). The element  $z_0$  can be taken from  $q^{\mathfrak{h} \ominus \mathfrak{h}'}$  while  $z_{\pm\alpha}$  have representatives in the Borel subalgebra  $U_q(\mathfrak{b}_{\pm})$ . The generators  $z_{\pm\alpha}$  can be expressed through extremal projectors [6, 7, 8] or alternatively as matrix entries of reduced Shapovalov inverse [9, 10] (the classical version appeared in [11, 12, 13]). In both cases, they require the rational extension,  $\hat{U}_q(\mathfrak{g})$ , of  $U_q(\mathfrak{g})$  over the ring of fractions  $\hat{U}_q(\mathfrak{h})$  of  $U_q(\mathfrak{h})$  with respect to a certain multiplicative system.

Regarding  $\hat{U}_q(\mathfrak{g})$  as a free right  $\hat{U}_q(\mathfrak{h})$ -module, one can think of  $z_{\pm\alpha}$  as rational trigonometric  $U_q(\mathfrak{g}_{\pm})$ -valued functions (raising and lowering operators) on  $\mathfrak{h}^*$ . Of course, they can be made polynomial upon multiplying by the common denominator of the Cartan coefficients. We call it natural regularization and denote the regularized generators by  $\check{z}_{\pm\alpha}$ . There arises the question whether  $\check{z}_{\pm\alpha}(\lambda) = 0$  at some  $\lambda$ . The answer is given in the present paper. We prove that for special linear and symplectic  $\mathfrak{g}$ ,  $\check{z}_{\pm\alpha}(\lambda) \neq 0$  for all  $\lambda \in \mathfrak{h}^*$ . For orthogonal  $\mathfrak{g}$ , there are  $\delta_{\pm\alpha} \in U_q(\mathfrak{h})$  such that  $\check{z}_{\pm\alpha}$  are divisible by  $\delta_{\pm\alpha}$  on the right and the quotient  $\check{z}_{\pm\alpha}\delta_{\pm\alpha}^{-1}$  does not turn zero at all weights.

Regularization of negative Mickelsson generators is more or less the same as regularization of singular vectors of  $V \otimes M_{\lambda}$  for  $V$  the "natural" representation of  $U_q(\mathfrak{g})$  and  $M_{\lambda}$  the Verma module of highest weight  $\lambda$ . In such a setting, it is a part of a more general problem when  $V$  is an arbitrary finite dimensional  $U_q(\mathfrak{g})$ -module. The analogous problem was considered for classical universal enveloping algebras in [14] and completely solved for  $\mathfrak{g} = \mathfrak{sl}(3)$ . We study the special case  $V = \mathbb{C}^N$  due to its significance. Remark that quantum groups lead to new effects which are absent in the classical version: the factors  $\delta_{\pm\alpha}$  turn to 1 for  $\mathfrak{g} = \mathfrak{so}(2n+1)$  in the limit  $q \rightarrow 1$ .

## 2 Quantum group preliminaries

Throughout the paper,  $\mathfrak{g}$  is a complex simple Lie algebra of type  $B$ ,  $C$  or  $D$ . Due to the natural inclusion  $U_q(\mathfrak{gl}(n)) \subset U_q(\mathfrak{g})$ , we do not pay special attention to this case (which was also covered in [15], Corollary 9.2, by different arguments). We fix a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  with the non-degenerate symmetric inner product  $(\cdot, \cdot)$  on  $\mathfrak{h}^*$ . By  $R$  we denote the root system of  $\mathfrak{g}$  with a fixed subsystem of positive roots  $R^+ \subset R$  and the basis of simple roots  $\Pi^+ \subset R^+$ . For every  $\lambda \in \mathfrak{h}^*$  we define its image  $h_\lambda$  under the isomorphism  $\mathfrak{h}^* \simeq \mathfrak{h}$ , that is  $(\lambda, \beta) = \beta(h_\lambda)$  for all  $\beta \in \mathfrak{h}^*$ . We denote by  $\rho$  the Weyl vector  $\frac{1}{2} \sum_{\alpha \in R^+} \alpha$ .

Suppose that  $q \in \mathbb{C}$  is not a root of unity. Denote by  $U_q(\mathfrak{g}_\pm)$  the  $\mathbb{C}$ -algebra generated by  $\{e_{\pm\alpha}\}_{\alpha \in \Pi^+}$ , subject to the  $q$ -Serre relations

$$\sum_{k=0}^{1-a_{\alpha\beta}} (-1)^k \begin{bmatrix} 1-a_{\alpha\beta} \\ k \end{bmatrix}_{q_\alpha} e_{\pm\alpha}^{1-a_{\alpha\beta}-k} e_{\pm\beta}^k = 0, \quad (2.1)$$

where  $a_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$  is the Cartan matrix,  $q_\alpha = q^{\frac{(\alpha, \alpha)}{2}}$ , and

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[k]_q! [m-k]_q!}, \quad [m]_q! = [1]_q \cdot [2]_q \cdot \dots \cdot [m]_q.$$

Here and further on,  $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$  whenever  $q^{\pm z}$  make sense.

Denote by  $U_q(\mathfrak{h})$  the commutative  $\mathbb{C}$ -algebra generated by  $q^{\pm h_\alpha}$ ,  $\alpha \in \Pi^+$ . The quantum group  $U_q(\mathfrak{g})$  is a  $\mathbb{C}$ -algebra generated by  $U_q(\mathfrak{g}_\pm)$  and  $U_q(\mathfrak{h})$  subject to the relations [16]

$$q^{h_\alpha} e_{\pm\beta} q^{-h_\alpha} = q^{\pm(\alpha, \beta)} e_{\pm\beta}, \quad [e_\alpha, e_{-\beta}] = \delta_{\alpha\beta} \frac{q^{h_\alpha} - q^{-h_\alpha}}{q_\alpha - q_\alpha^{-1}}. \quad (2.2)$$

Although  $\mathfrak{h}$  is not contained in  $U_q(\mathfrak{g})$ , still it is convenient to keep reference to  $\mathfrak{h}$ .

Fix the comultiplication in  $U_q(\mathfrak{g})$  as in [17]:

$$\begin{aligned} \Delta(e_\alpha) &= e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha, & \Delta(e_{-\alpha}) &= e_{-\alpha} \otimes 1 + q^{-h_\alpha} \otimes e_{-\alpha}, \\ \Delta(q^{\pm h_\alpha}) &= q^{\pm h_\alpha} \otimes q^{\pm h_\alpha}, \end{aligned}$$

for all  $\alpha \in \Pi^+$ .

The subalgebras  $U_q(\mathfrak{b}_\pm) \subset U_q(\mathfrak{g})$  generated by  $U_q(\mathfrak{g}_\pm)$  over  $U_q(\mathfrak{h})$  are quantized universal enveloping algebras of the Borel subalgebras  $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{g}_\pm \subset \mathfrak{g}$ . The multiplication map implements an isomorphism  $U_q(\mathfrak{g}_-) \otimes U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{h}) \rightarrow U_q(\mathfrak{g})$  of vector spaces, which also descends to the decomposition  $U_q(\mathfrak{b}_\pm) = U_q(\mathfrak{g}_\pm) U_q(\mathfrak{h})$ .

The notation  $n$  is reserved for the rank of  $\mathfrak{g}$ . We enumerate the elements of  $\Pi^+$  so that  $\mathfrak{gl}(n)$  is a Lie subalgebra in  $\mathfrak{g}$  with simple roots  $\{\alpha_i\}_{i=1}^{n-1}$ , and  $U_q(\mathfrak{gl}(n))$  the corresponding quantum subgroup in  $U_q(\mathfrak{g})$ . Let  $\mathfrak{gl}(s) \subset \mathfrak{gl}(n)$  be the maximal subalgebra stable under automorphisms of the Dynkin diagram of  $\mathfrak{g}$ . One has  $s = n$  for  $\mathfrak{g} = \mathfrak{sp}(2n), \mathfrak{so}(2n+1)$  and  $s = n-1$  for  $\mathfrak{g} = \mathfrak{so}(2n)$ .

We use the notation  $e_i = e_{\alpha_i}$  and  $f_i = e_{-\alpha_i}$  for  $\alpha_i \in \Pi^+$  in all cases apart from  $i = n$ ,  $\mathfrak{g} = \mathfrak{so}(2n+1)$ , when we set  $f_n = [\frac{1}{2}]_q e_{-\alpha_n}$ . This modifies the relation (2.2) to

$$[e_n, f_n] = \frac{q^{h_{\alpha_n}} - q^{-h_{\alpha_n}}}{q - q^{-1}}.$$

All other relations stay intact.

## 2.1 Natural representation

In this section we recall the natural representation of  $\mathfrak{g}$  in the vector space  $\mathbb{C}^N$ . Let  $\{w_i\}_{i=1}^N$  be the standard basis in  $\mathbb{C}^N$ . We used the notation  $i' = N+1-i$  for all integers  $i \in I = [1, N]$  corresponding to the flip of the integer interval  $I$  around the center  $\frac{N+1}{2}$ . To improve readability of formulas, we use special notation  $* = \frac{N+1}{2}$ .

The natural representation is constructed as follows. We assign the matrices

$$\pi(e_i) = E_{i,i+1} + E_{i'-1,i'}, \quad \pi(f_i) = E_{i+1,i} + E_{i',i'-1}, \quad \pi(h_{\alpha_i}) = E_{ii} - E_{i+1,i+1} + E_{i'-1,i'-1} - E_{i'i'},$$

for  $i = 1, \dots, n-1$ . This defines a direct sum of two representations of the subalgebra  $U_q(\mathfrak{gl}(n))$ . We extend it to the representation of  $U_q(\mathfrak{g})$  as

$$\pi(e_n) = E_{n,*} + E_{n'-1,n'}, \quad \pi(f_n) = E_{*,n} + E_{n',*}, \quad \pi(h_{\alpha_n}) = E_{nn} - E_{n'n'},$$

$$\pi(e_n) = E_{nn'}, \quad \pi(f_n) = E_{n'n}, \quad \pi(h_{\alpha_n}) = 2E_{nn} - 2E_{n'n'},$$

$$\pi(e_n) = E_{n-1,n'} + E_{n,n'+1}, \quad \pi(f_n) = E_{n',n-1} + E_{n'+1,n}, \quad \pi(h_{\alpha_n}) = E_{n-1,n-1} + E_{nn} - E_{n'n'} - E_{n'+1,n'+1},$$

respectively, for  $\mathfrak{g} = \mathfrak{so}(2n+1)$ ,  $\mathfrak{g} = \mathfrak{sp}(2n)$ , and  $\mathfrak{g} = \mathfrak{so}(2n)$ . The Cartan subalgebra is represented by diagonal matrices, and the basis elements  $w_i$  carry weights  $\varepsilon_i \in \mathfrak{h}^*$  with  $\varepsilon_{i'} = -\varepsilon_i$ . The set  $\{\varepsilon_i\}_{i=1}^n$  forms an orthonormal basis  $\mathfrak{h}^*$ .

We introduce a partial ordering on the integer interval  $[1, N]$  by setting  $i \preccurlyeq j$  if and only if  $w_j \in U_q(\mathfrak{g}_-)w_i$ . One has  $i \prec j \Rightarrow i < j$ .

Define a matrix  $F \in \text{End}(V) \otimes U_q(\mathfrak{g}_-)$  as  $F = (\pi \otimes \text{id})(q^{-\sum_{i=1}^n h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \mathcal{R})$ . Its entries  $f_{ij}$  are expressed through modified commutators  $[x, y]_a = xy - ayx$ ,  $a \in \mathbb{C}$ , as given below. For

all  $\mathfrak{g}$  and  $i < j \leq * \text{ set}$

$$f_{ij} = [f_{j-1}, \dots [f_{i+1}, f_i]_{\bar{q}} \dots]_{\bar{q}}, \quad f_{j'i'} = [\dots [f_i, f_{i+1}]_{\bar{q}}, \dots f_{j-1}]_{\bar{q}}. \quad (2.3)$$

where  $f_{i,i+1} = f_i = f_{i'-1,i}$  is understood. Here and further on the bar designates the inverse, e.g.  $\bar{q} = q^{-1}$ . Furthermore,

- for  $\mathfrak{g} = \mathfrak{so}(2n+1)$ :  $f_{nn'} = (q-1)f_n^2$  and

$$f_{ij'} = q^{\delta_{ij}} [f_{*,j'}, f_{i,*}]_{\bar{q}^{\delta_{ij}}}, \quad i, j < n. \quad (2.4)$$

- for  $\mathfrak{g} = \mathfrak{sp}(2n)$ :  $f_{nn'} = [2]_q f_n$  and

$$f_{in'} = [f_n, f_{in}]_{\bar{q}^2}, \quad f_{ni'} = [f_{n'i'}, f_n]_{\bar{q}^2}, \quad i < n, \quad (2.5)$$

$$f_{ij'} = q^{\delta_{ij}} [f_{nj'}, f_{in}]_{\bar{q}^{1+\delta_{ij}}}, \quad i, j < n. \quad (2.6)$$

- for  $\mathfrak{g} = \mathfrak{so}(2n)$ :  $f_{nn'} = 0$  and

$$f_{in'} = [f_n, f_{i,n-1}]_{\bar{q}}, \quad f_{ni'} = [f_{n'+1,i'}, f_n]_{\bar{q}}, \quad i < n-2, \quad (2.7)$$

$$f_{j'i'} = q^{\delta_{ij}} [f_{ni'}, f_{j,n}]_{\bar{q}^{1+\delta_{ij}}}, \quad i, j < n. \quad (2.8)$$

Finally,  $f_{ii} = 1$  for all  $i$  and  $f_{ij} = 0$  for  $i > j$ .

The matrix  $F$  participates in construction of reduced Shapovalov inverse form  $\hat{F} = \sum_{i,j=1}^N E_{ij} \otimes \hat{f}_{ij}$ , which is given next [9]. It is convenient to use the language of Hasse diagram of the  $\prec$ -poset  $I$ , whose arcs are labeled with negative Chevalley generators (directed toward superior nodes). We call any ascending sequence of nodes  $(m_i)_{i=1}^k \subset I$  a route from  $m_1$  to  $m_k$ . A maximal route, i.e. whose adjacent nodes are connected with arcs, is called path. If  $i \prec m_1$  and  $m_k \prec j$  for some  $i, j$ , we write  $i \prec \vec{m} \prec j$ .

For all  $i, j \in I$  define  $\eta_{ij} \in \mathfrak{h} + \mathbb{C}$  by

$$\eta_{ij} = h_{\varepsilon_i} - h_{\varepsilon_j} + \rho_i - \rho_j - \frac{1}{2} \|\varepsilon_i - \varepsilon_j\|^2,$$

where  $\rho_i = (\rho, \varepsilon_i)$ , and  $\|\beta\|^2$  is the Euclidean norm of  $\beta \in \mathfrak{h}^*$ . We regard  $\eta_{ij}$  as an affine function  $\mathfrak{h}^* \rightarrow \mathbb{C}$ ,  $\lambda \mapsto (\lambda + \rho, \varepsilon_i - \varepsilon_j) - \frac{1}{2} \|\varepsilon_i - \varepsilon_j\|^2$ . The entries  $\hat{f}_{ij}$  are constructed as

follows. Put  $\hat{f}_{ii} = 1$  and  $\hat{f}_{ij} = 0$  for  $i > j$ . For  $i < j$ , define  $A_i^j = \frac{q - q^{-1}}{q^{-2\eta_{ij}} - 1}$ . For a route  $\vec{m}$  denote  $f_{\vec{m}} = f_{m_1, m_2} \cdots f_{m_{k-1}, m_k}$  and  $A_{\vec{m}}^j = A_{m_1}^j \cdots A_{m_k}^j$ . Then

$$\hat{f}_{ij} = \sum_{i \prec \vec{m} \prec j}^{\emptyset} f_{i, \vec{m}, j} A_{i, \vec{m}}^j,$$

where the symbol  $\emptyset$  indicates here and further on that the empty route  $\vec{m} = \emptyset$  is included. The elements  $\hat{f}_{1j}$ , where  $j$  ranges from 2 to  $N$  for  $\mathfrak{g} = \mathfrak{sp}(N)$  and to  $N - 1$  for  $\mathfrak{g} = \mathfrak{so}(N)$  form the set of negative generators of  $Z_q(\mathfrak{g}, \mathfrak{g}')$  where  $\mathfrak{g}' \subset \mathfrak{g}$  is the simple Lie subalgebra with the root basis  $\{\alpha_2, \dots, \alpha_n\}$ .

Let  $M_\lambda$  be the Verma module of highest weight  $\lambda \in \mathfrak{h}^*$  with the canonical generator  $v_\lambda$ . The matrix  $\hat{F}$  is regarded as a map  $\mathfrak{h}^* \mapsto \text{End}(V) \otimes U_q(\mathfrak{g}_-)$ , such that  $\hat{f}_{ij}(\lambda)v_\lambda = \hat{f}_{ij}v_\lambda$ . The tensors  $\hat{F}_j = \hat{F}(w_j \otimes v_\lambda) = \sum_{i=1}^j w_i \otimes \hat{f}_{ij}v_\lambda$  are singular vectors, i.e. annihilated by all  $e_\alpha$ ,  $\alpha \in \Pi^+$ . They are well defined for generic  $\lambda$  and generate submodules  $M_j \simeq M_{\lambda + \varepsilon_j} \subset V \otimes M_\lambda$ . At some weights,  $\hat{F}_j$  have zeros and poles and therefore need a regularization, as singular vectors are defined up to a scalar multiplier. In particular, there is a natural regularization  $\check{f}_{ij} = \hat{f}_{ij} \prod_{l \prec j} \bar{A}_l \in U_q(\mathfrak{b}_-)$ . It turns out to be excessive in some cases as having zeros at some  $\lambda$ . We study this issue for the most important pairs  $(i, j)$  relative to the generators of  $Z_q(\mathfrak{g}, \mathfrak{g}')$ .

### 3 Standard filtration in $V \otimes M_\lambda$

Our study of the matrix  $\hat{F}$  is based on analysis of the tensor product  $V \otimes M_\lambda$ . To a large extent, its module structure is captured by a  $U_q(\mathfrak{g})$ -invariant operator  $\mathcal{Q} = (\pi \otimes \text{id})(\mathcal{R}_{21}\mathcal{R}) \in \text{End}(V) \otimes M_\lambda$ , which is scalar on highest weight submodules and factor modules. Denoting by  $x_j$  its eigenvalue on the submodule  $M_j$ , one has  $x_i x_j^{-1} = q^{2\xi_{ij}}|_\lambda$  with

$$\xi_{ij} = h_{\varepsilon_i} - h_{\varepsilon_j} + \rho_i - \rho_j + \frac{1}{2}(|\varepsilon_i|^2 - |\varepsilon_j|^2).$$

For generic  $\lambda$  the eigenvalues  $x_j$  are pairwise distinct and separate  $M_j$ .

Another tool for the analysis of  $V \otimes M_\lambda$  is the sequence of submodules  $(V_j)_{j=1}^N$  generated by  $v_{\lambda, l} = w_l \otimes v_\lambda$ ,  $l = 1, \dots, j$ . It forms an ascending filtration of  $V \otimes M_\lambda$  whose graded module is isomorphic to the direct sum  $\bigoplus_{j=1}^N V_j/V_{j-1}$  of Verma modules. This implies that  $V_j$  are all  $\mathcal{Q}$ -invariant, [19]. In the present section we study projection  $\wp_j: M_j \rightarrow V_j/V_{j-1}$ , which can be either zero or an isomorphism. By weight arguments,  $M_j \subset V_j$ , and the singular vector  $\hat{F}_j$  is mapped into the line generated  $v_{\lambda, j} \bmod V_{j-1}$ , the highest vector of  $V_j/V_{j-1}$ .

Define  $\hat{C}_i$  by  $\wp_j(\hat{F}_j) = \hat{C}_j v_{\lambda, j}$ . Our nearest goal is computation of  $\hat{C}_j$ ,  $j = 1, \dots, N$ . It is clear that  $\hat{C}_1 = 1$ . For  $j > 1$  the answer is given by Proposition 3.2 below.

For all  $j \in I$  introduce a commutative algebra  $\mathcal{A}_j$  as follows. For  $j \leq \frac{N+1}{2}$  set  $\mathcal{A}_j = \mathbb{C}[y_1^{\pm 1}, \dots, y_{j-1}^{\pm 1}]$ . Otherwise put  $\mathcal{A}_j$  to be the quotient of  $\mathbb{C}[y_1^{\pm 1}, \dots, y_{j-1}^{\pm 1}]$  modulo the relations  $y_l y_{l'} = y_{j'}$ ,  $l = j' + 1, \dots, j - 1$ , and, for  $\mathfrak{g} = \mathfrak{so}(2n + 1)$ , extended with  $y_{j'}^{\frac{1}{2}}$  subject to  $q y_* = y_{j'}^{\frac{1}{2}}$ . In all cases,  $\mathcal{A}_j$  is a localization of a polynomial algebra. We can realize  $\mathcal{A}_j$  as a subalgebra in  $\hat{U}_q(\mathfrak{h})$  via the assignment  $y_l = q^{-2\eta_j}$ ,  $l \prec j$ , due to the following fact.

**Proposition 3.1.** *For all  $m = 2, \dots, n$  one has  $\eta_{m1'} + \eta_{m'1'} = \eta_{11'}$ . Also, for  $\mathfrak{g} = \mathfrak{so}(2n + 1)$ ,  $2\eta_{*, m'} - 1 = \eta_{m, m'}$ ,  $m = 1, \dots, n$ .*

*Proof.* Straightforward. □

We regard  $\hat{f}_{ij}$  as an element of the free  $\mathcal{A}_j$ -module generated by  $U_q(\mathfrak{g}_-)$ .

Consider  $\hat{C}_j$  as a polynomial in  $B_i = -A_i$ ,  $i = 1, \dots, j - 1$ , where  $A_i = \frac{q - \bar{q}}{y_i - 1}$ . Let  $|i - j|$  denote the number of arcs in a path from  $i$  to  $j$  on the Hasse diagram

**Proposition 3.2.** *The coefficients  $\hat{C}_j$ ,  $j = 2, \dots, N$ , factorize as*

$$\hat{C}_j = \begin{cases} (1 - [2]_q q^2 A_j) \prod_{\substack{i=1, \\ i \neq j'}}^{j-1} (1 - q A_i) & \mathfrak{g} = \mathfrak{sp}(N), \\ \frac{y_* - q}{y_* - \bar{q}} \prod_{\substack{i=1, \\ i \neq j', *}}^{j-1} (1 - q A_i), & \mathfrak{g} = \mathfrak{so}(2n + 1), \\ \prod_{\substack{i=1, \\ i \neq j'}}^{j-1} (1 - q A_i), & \mathfrak{g} = \mathfrak{so}(2n), \end{cases}$$

where factor  $\frac{y_* - q}{y_* - \bar{q}}$  is present only when  $*$   $<$   $j$ .

The proof is based on the concept of principal monomial, which is associated with every pair  $i \prec j$ . Each path from  $i$  to  $j$  gives rise to a unique element  $\psi_{ji} \in U_q(\mathfrak{g}_-)$  such that  $w_j = \psi_{ji} w_i$ . We denote by  $\psi^{ij}$  the element obtained from  $\psi_{ji}$  by reverting the order of simple factors and call it principal monomial of the pair  $(i, j)$ .

**Lemma 3.3** ([19]). *Suppose that  $i \prec j$  and  $\psi \in U_q(\mathfrak{g}_-)$  is a Chevalley monomial of weight  $\varepsilon_j - \varepsilon_i$  distinct from  $\psi^{ij}$ . Then  $\wp_j(\psi) = 0$ . Furthermore  $\wp_j(w_i \otimes \psi^{ij} v_\lambda) = (-1)^{|i-j|} q^{\tilde{\rho}_i - \tilde{\rho}_j} v_{\lambda, j}$ , where  $\tilde{\rho}_i = \rho_i + \frac{1}{2} \|\varepsilon_i\|^2$ .*

*Proof.* The first part of the statement is proved in [19], Lemma 3.4. Similar statement for a different version of the comultiplication is also proved therein. Here we give a proof for the current version of the quantum group. Suppose that  $\alpha \in \Pi^+$  and  $\varepsilon_i - \varepsilon_k = \alpha$ . By [19], Lemma 3.4, the node  $w_i \otimes \psi^{kj} v_\lambda$  lies in  $V_{j-1}$ . Applying  $\Delta f_\alpha = f_\alpha \otimes 1 + q^{-h_\alpha} \otimes f_\alpha$  to  $w_i \otimes \psi^{kj} v_\lambda$  we get

$$w_i \otimes \psi^{ij} v_\lambda = -q^{(\alpha, \varepsilon_i)} w_k \otimes \psi^{kj} v_\lambda = -q^{\tilde{\rho}_i - \tilde{\rho}_k} w_k \otimes \psi^{kj} v_\lambda \pmod{V_{j-1}} \quad (3.9)$$

for all  $k \prec j$ . Here we used  $f_\alpha w_i = w_k$  and  $f_\alpha \psi^{kj} = \psi^{ij}$  for all  $k \prec j$ . Proceeding along the path from  $i$  to  $j$  we complete the proof.  $\square$

Thanks to Lemma 3.3, only the principal term of  $\hat{f}_{ij}$  contributes to  $\wp_j(w_j \otimes v_\lambda)$ . Put  $\sigma = 1$  for symplectic  $\mathfrak{g}$  and  $\sigma = -1$  for orthogonal  $\mathfrak{g}$ .

**Lemma 3.4.** *The  $(i, j)$ -principal term of  $f_{ij}$  is  $(-1)^{|i-j|-1} c_{ij} \psi^{ij}$  with*

$$c_{ij} = \begin{cases} \bar{q}^{\eta_{ij}(0)}, & i \neq j', \\ \bar{q} \bar{q}^{\eta_{ij}(0)} + \sigma q, & i = j'. \end{cases}$$

*Proof.* Straightforward.  $\square$

Given a route  $\vec{m} = (m_1, \dots, m_k)$  put  $c_{\vec{m}} = c_{m_1, m_2} \dots c_{m_{k-1}, m_k}$ . Introduce  $\hat{c}_{ij} \in \mathbb{C}$  via the equality  $\wp_j(w_i \otimes \hat{f}_{ij} v_\lambda) = \hat{c}_{ij} v_{\lambda, j}$ , so that  $\hat{C}_j = \sum_{i=1}^j \hat{c}_{ij}$ . It is easy to check  $\hat{c}_{ij} = \sum_{i \prec \vec{m} \prec j} c_{i, \vec{m}, j} B_{i, \vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_j}$ .

*Proof of Proposition 3.2 for  $j = N$ .* We prove this special case by induction on  $N$ . The base for induction is immediate  $\hat{C}_2 = 1$  for  $\mathfrak{g} = \mathfrak{so}(2)$ . It is less obvious although straightforward  $\hat{C}_3 = \frac{y^* - q}{y^* - \bar{q}}$  for  $\mathfrak{g} = \mathfrak{so}(3)$  and  $\hat{C}_2 = 1 - [2]_q q^2 A_1$  for  $\mathfrak{g} = \mathfrak{sp}(2)$ . So we assume  $\varepsilon_2 \neq 0$  for higher  $N$  in what follows. Our strategy is to factor out  $\phi_i = B_i + q^{-1}$  for  $i = 2', 2$ . This will facilitate the induction transition.

With  $1 + \hat{c}_{2'1'} = 1 + B_{2'} q^{\tilde{\rho}_{2'} - \tilde{\rho}_{1'}} = q \phi_{2'}$ , let us calculate  $\sum_{i=3}^{2'} \hat{c}_{i1'}$  next. Observe that for all  $3 \prec \vec{m} \prec 2'$  we have the equality  $c_{i, \vec{m}, 1'} = q^{-1} c_{i, \vec{m}, 2', 1'} = q^{-1} c_{i, \vec{m}, 2'}$ . Also, for all  $i$  we replace  $q^{\tilde{\rho}_i - \tilde{\rho}_{1'}} = q q^{\tilde{\rho}_i - \tilde{\rho}_{2'}}$ . Then

$$\begin{aligned} 1 + \sum_{i=3}^{2'} \hat{c}_{i1'} &= q \phi_{2'} + \sum_{i=3}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i, \vec{m}, 2', 1'} B_{i, \vec{m}} B_{2'} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}} + \sum_{i=3}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i, \vec{m}, 1'} B_{i, \vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}}. \\ &= q \phi_{2'} (1 + \sum_{i=3}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i, \vec{m}, 2'} B_{i, \vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{2'}}). \end{aligned}$$



The sum  $\sum_{i=1}^2 \hat{c}_{i1'}$  reads

$$\begin{aligned} &= \sum_{i=1}^2 \sum_{i \prec \vec{m} \prec 1'}^{\emptyset} c_{i,\vec{m},1'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}} = \sum_{i=1}^2 \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} (c_{i,\vec{m},2',1'} B_{i,\vec{m},2'} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}} + c_{i,\vec{m},1'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}}) \\ &= \phi_{2'} \sum_{i=1}^2 \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i,\vec{m},2'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}} + \sum_{i=1}^2 \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} (c_{i,\vec{m},1'} - q^{-1} c_{i,\vec{m},2'}) B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{1'}}. \end{aligned}$$

The only nonzero differences in the last sum correspond to  $i = 1, \vec{m} = \emptyset, \vec{m} = (2)$  and  $i = 2, \vec{m} = \emptyset$ . They are equal to  $c_{1,1'} - q^{-1} c_{1,2'} = q^{-\theta_{1,1'} - 1} + \sigma q - q^{-1} q^{-\theta_{1,2'}} = \sigma q$  and  $c_{1,2,1'} - q^{-1} c_{1,2,2'} = c_{2,1'} - q^{-1} c_{2,2'} = q^{-\theta_{2,1'}} - \sigma - q^{-1} q^{-\theta_{2,2'} - 1} = -\sigma$ , respectively. This gives the last term

$$\sigma q (q B_1 - B_{1,2} - B_2 q^{\tilde{\rho}_2 - \tilde{\rho}_1}) q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}} = \sigma q \phi_{2'} (B_1 - B_2) q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}},$$

since  $\tilde{\rho}_2 - \tilde{\rho}_1 = \rho_2 - \rho_1 = -1$ . This way we factor out  $q \phi_{2'}$  in the expression for  $\hat{C}_{1'}$ :

$$\frac{\hat{C}_{1'}}{q \phi_{2'}} = 1 + \sum_{i=1}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i,\vec{m},2'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{2'}} + \sigma (B_1 - B_2) q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}}. \quad (3.10)$$

Consider separately the sums over  $i = 1, 2$  and  $3 \leq i \leq 3'$ . First, using the equality  $c_{12'} - q^{-1} c_{22'} = q^{-\theta_{1,2'}} - \sigma - q^{-\theta_{2,2'} - 2} = -\sigma$ , develop the internal sum for  $i = 1$  as

$$\sum_{2 \prec \vec{m} \prec 2'}^{\emptyset} (c_{1,2,\vec{m},2'} B_{1,\vec{m}} B_2 + c_{1,\vec{m},2'} B_{1,\vec{m}}) q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}} = q \phi_2 \sum_{2 \prec \vec{m} \prec 2'}^{\emptyset} c_{2,\vec{m},2'} B_{1,\vec{m}} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} - \sigma B_1 q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}},$$

where the substitution  $q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}} = q q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}}$  is made.

Next observe that for  $3 \leq k \leq 3'$  and  $l = 1, 2$ , we have  $c_{lk} = q^{-\theta_{lk}} = q^{-\rho_l + \rho_k + \frac{\|\varepsilon_l - \varepsilon_k\|^2}{2}} = q^{-\tilde{\rho}_l + \tilde{\rho}_k + \frac{\|\varepsilon_l - \varepsilon_k\|^2}{2} + \frac{\|\varepsilon_l\|^2 - \|\varepsilon_k\|^2}{2}} = q^{-\tilde{\rho}_l + \tilde{\rho}_k + 1}$ , since  $\varepsilon_2 \neq 0$  by the assumption. Rewrite the internal sum for  $i = 2$  in (3.10) as

$$c_{22'} B_2 q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} + \sum_{k=3}^{3'} \sum_{k \prec \vec{m} \prec 2'}^{\emptyset} c_{k,\vec{m},2'} B_{k,\vec{m}} B_2 q^{\tilde{\rho}_k - \tilde{\rho}_{2'} + 1}.$$

Along with the sum over  $3 \leq i \leq 3'$  in (3.10), this gives

$$= -c_{22'} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} + \phi_2 c_{22'} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} + \phi_2 \sum_{i=3}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i,\vec{m},2'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{2'} + 1}.$$

Upon these transformations, (3.10) turns into

$$\begin{aligned} \frac{\hat{C}_{1'}}{q \phi_{2'}} &= 1 - c_{22'} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'} - 1} - \sigma (\phi_2 - q^{-1}) q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}} + \phi_2 c_{22'} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} \\ &+ q \phi_2 \sum_{i=3}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i,\vec{m},2'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{2'}} + q \phi_2 \sum_{2 \prec \vec{m} \prec 2'}^{\emptyset} c_{2,\vec{m},2'} B_{1,\vec{m}} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}}. \end{aligned}$$

Since  $-\sigma q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}} + c_{22'} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} = -\sigma q^{\tilde{\rho}_1 - \tilde{\rho}_{2'}} + (q^{-\theta_{22'} - 1} + \sigma q) q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} = q^{-\theta_{22'} - 1} q^{\tilde{\rho}_2 - \tilde{\rho}_{2'}} = q$ , the first line gives  $q\phi_2$ . Eventually, we get

$$\hat{C}_{1'}/\phi_2\phi_{2'}q^2 = 1 + \sum_{i=2}^{3'} \sum_{i \prec \vec{m} \prec 2'}^{\emptyset} c_{i,\vec{m},2'} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_{2'}}.$$

The right-hand side is exactly  $\hat{C}_{2'}(y_1, y_3, \dots, y_{3'})$  for  $\dim V = N - 2$ . This yields  $\hat{C}_{1'}$  by induction on  $N$ .  $\square$

*Proof of Proposition 3.2 for the general  $j$ .* Now we assume  $1 < j' \leq j$ . Let  $k$  be  $j + 1$ , so that  $k' = j' - 1$ . Using the formula  $c_{1i} = q^{\tilde{\rho}_i - \tilde{\rho}_1 + 1}$  for  $1 < i < j' - 1$ , present  $\hat{c}_{1j} = \sum_{1 \prec \vec{m} \prec j'}^{\emptyset} c_{1,\vec{m}} B_{1,\vec{m}} q^{\tilde{\rho}_1 - \tilde{\rho}_j}$  as

$$c_{1j} q^{\tilde{\rho}_1 - \tilde{\rho}_j} + B_1 \sum_{i=2}^{j-1} \sum_{i \prec \vec{m} \prec j}^{\emptyset} c_{1i} c_{i,\vec{m}} B_{i,\vec{m}} q^{\tilde{\rho}_1 - \tilde{\rho}_j} = q B_1 (1 + \sum_{i=2}^{j-1} \sum_{i \prec \vec{m} \prec j'}^{\emptyset} c_{i,\vec{m}} B_{i,\vec{m}} q^{\tilde{\rho}_i - \tilde{\rho}_j}).$$

This is equal to  $q B_1 (1 + \sum_{i=2}^{j-1} \hat{c}_{ij})$ .  $\hat{C}_j = 1 + \sum_{i=2}^{j-1} \hat{c}_{ij} + \hat{c}_{1j} = q B_1 (1 + \sum_{i=2}^{j-1} \hat{c}_{ij})$ . Induction gives  $\hat{C}_j = q B_1 \dots q B_{j'-1} (1 + \sum_{i=j'}^{j-1} \hat{c}_{ij})$ . The expression in the bracket is nothing but  $\hat{C}_j(y_{j'}, y_{j'+1}, \dots, y_{j-1})$  for  $\dim V = j$ . Its factorization is already proved.  $\square$

We apply Proposition 3.2 for the analysis of the matrix entries  $\check{f}_{ij}$  specifically for orthogonal  $\mathfrak{g}$ . Introduce  $\delta_j^- \in \mathcal{A}_j$  as follows. Put  $\delta_j^- = 1$  for  $j < s'$  and, for  $j \geq s'$ ,  $\delta_j^- = \frac{y_{j'} - 1}{q - \bar{q}} = \bar{A}_{j'}$  if  $\mathfrak{so}(2n)$  and  $\delta_j^- = \frac{y_{j'}^{\frac{1}{2}} + 1}{q + 1}$  if  $\mathfrak{so}(2n + 1)$ . For symplectic  $\mathfrak{g}$ , set  $\delta_j^- = 1$  for all  $j$ .

**Lemma 3.5.** *Suppose that  $\mathfrak{g} = \mathfrak{so}(N)$ . For all  $i, j$  such that and  $i \leq j'$  the element  $\check{f}_{ij}$  is divisible by  $\delta_j^-$ .*

*Proof.* Only the case  $i \leq s, s' \leq j$  requires consideration. Without loss of generality, we can set  $i = 1$ . Recall the basis decomposition  $\hat{F}_j = \sum_{l=1}^j w_l \otimes \hat{f}_{lj} v_\lambda$  and consider the singular vector  $\hat{F}_j^\# = \frac{y_{j'} - 1}{q - \bar{q}} \hat{F}_j$ . Under the specialization  $q^{-2\eta_{l1}}|_\lambda = y_l, l \prec j$ , we have  $\hat{f}_{1j}(\lambda) = \hat{f}_{1j}^\# \frac{q - \bar{q}}{y_{j'} - 1}$ . Observe that  $y_{j'} - 1$  is divisible by  $\delta_j^-$ , and  $A_{j'} = \frac{q - \bar{q}}{y_{j'} - 1}$  is the only factor in  $\hat{f}_{1j}$  that has a pole at  $\delta_j^- = 0$ , so  $\hat{F}_j^\#$  is regular at  $\delta_j^- = 0$ . Since  $\hat{C}_j$  is regular at generic  $\lambda$  subject to  $\delta_j^- = 0$ , one has  $\wp_j(\hat{F}_j^\#) = 0$ . This is possible only if either  $\hat{F}_j^\# = 0$  or  $\hat{F}_j^\# \in V_{j-1}$ . Of all  $\mathcal{Q}$ -eigenvalues only  $x_j, x_{j'}$  for even  $N$  and  $x_j, x_*, x_{j'}$  for odd  $N$  become constant at  $\delta_j^- = 0$ , while the other can be made distinct from  $x_j$ . Since  $x_j x_{j'}^{-1} = q^{-4} y_{j'} \neq 1$  and  $x_j x_*^{-1} = q^{-1} y_{j'}^{\frac{1}{2}} \neq 1$ , the spectrum of  $\mathcal{Q}$  restricted to  $V_{j-1}$  does not contain  $x_j$ , its eigenvalue on  $M_j$ . Therefore  $\hat{F}_j^\#$  vanishes along with its coefficient  $\hat{f}_{1j}^\#$ , subject to  $\delta_j^- = 0$ . Hence  $\check{f}_{1j}$  is divisible by  $\delta_j^-$ .  $\square$

## 4 Regularization of matrix coefficients

In this section we describe regularization for  $\hat{f}_{ij}$ . We consider reductions of  $\check{f}_{ij}$  along two-sided ideals in  $\mathcal{B} = U_q(\mathfrak{g}_-)$  from the following family. Denote by  $\mathfrak{S}$  the union of integer intervals  $[1, s] \cup [s', 1']$  and by  $\mathfrak{S}_{ij} = ]i, j[ \cap \mathfrak{S}$  for all pairs  $i \prec j$ . Fix  $k \in \mathfrak{S}$  and let  $l, r$  be the nearest nodes to  $k$  such that  $l \prec k \prec r$ . Suppose that  $f_\alpha$  and  $f_\beta$  corresponding to  $\alpha = \varepsilon_k - \varepsilon_l$  and  $\beta = \varepsilon_r - \varepsilon_k$  do not commute. We define the two sided ideal  $J_k \subset \mathcal{B}$  generated by all such  $f_{lr}$  (exactly one unless  $\mathfrak{g} = \mathfrak{so}(2n)$  and  $k = s, s'$ ). As follows from (2.1),  $\mathcal{B}/J_k$  is isomorphic as a vector space to  $U_q(\mathfrak{g}_-^1) \otimes U_q(\mathfrak{g}_-^2)$ , where  $\mathfrak{g}^i \subset \mathfrak{g}$  are certain Lie subalgebras. One has the relation  $f_\beta f_\alpha = \bar{q} f_\alpha f_\beta$  in all cases apart from  $\mathfrak{g} = \mathfrak{sp}(2n)$  and  $k = n, n'$ ; then  $f_\beta f_\alpha = \bar{q}^2 f_\alpha f_\beta$ . Furthermore

**Lemma 4.1.** *For all  $i \prec j$  and  $k \in \mathfrak{S}_{ij}$ , one has  $f_{ij} = 0$  modulo  $J_k$  and  $f_{ij} = (q - \bar{q})f_{ik}f_{kj}$  modulo  $J_{k'}$ .*

*Proof.* Readily follows from (2.3-2.8). □

**Corollary 4.2.** *Suppose that  $i \prec j$ . Then*

- i) *for all  $k \in \mathfrak{S}_{ij}$ ,  $\check{f}_{ij} = \check{f}_{ik}\check{f}_{kj} \pmod{J_k}$  and  $\check{f}_{ij} = \check{f}_{ik}\check{f}_{kj}y_k \pmod{J_{k'}}$ ,*
- ii) *if  $k, k' \in \mathfrak{S}_{ij}$  and  $k < k'$ , then  $\check{f}_{ij} = \check{f}_{ik}\check{f}_{kk'}\check{f}_{k'j}y_{k'} \pmod{J_k}$ .*

*Proof.* Let us check i) first. Fix  $k \in \mathfrak{S}_{ij}$  and write

$$\hat{f}_{ij} = \sum_{i \prec \vec{m} \prec k \prec \vec{l} \prec j}^{\emptyset} (f_{i, \vec{m}, k} A_{i, \vec{m}} f_{k, \vec{l}, j} A_{\vec{l}} A_k + f_{i, \vec{m}, \vec{l}, j} A_{i, \vec{m}} A_{\vec{l}}).$$

The second sum vanishes modulo  $J_k$ , and this implies the first factorization. The element  $f_{i, \vec{m}, \vec{l}, j}$  factorizes as  $(q - q^{-1})f_{i, \vec{m}, k} f_{k, \vec{l}, j}$  modulo  $J_{k'}$  in accordance with Lemma 4. Taking into account that  $(A_k + q - \bar{q})\bar{A}_k = y_k$ , we get factorization modulo  $J_{k'}$ . Finally, observe that ii) follows from i). □

**Corollary 4.3.** *The elements  $\check{f}_{ij}(\lambda)$  with  $i \prec j < s'$  or  $s < i \prec j$  do not turn zero at all  $\lambda$ .*

*Proof.* Indeed, projecting  $\check{f}_{ij}$  along  $\sum_{k \in \mathfrak{S}_{ij}} J_k$  we get  $f_{\vec{m}}$ , where  $\vec{m}$  is the (unique) path from  $i$  to  $j$ . All factors in the product are Shevalley generators, therefore  $f_{\vec{m}}$  and hence  $\check{f}_{ij}$  do not vanish. □

**Theorem 4.4.** *Suppose that  $i \prec j$  and  $i \leq j'$ . Then  $\check{f}_{ij}(\lambda)/\delta_j^-(\lambda) \neq 0$  at all  $\lambda \in \mathfrak{h}^*$ .*

*Proof.* For  $\mathfrak{g} = \mathfrak{sp}(2n)$  we have  $s = n$  and  $\delta_j^- = 1$ . In view of Lemma 4.3, it is sufficient to check the case  $i \leq n$ ,  $n' \leq j$ . Taking projection modulo  $J_s$  we get factorization  $\check{f}_{ij} = \check{f}_{in}\check{f}_{nn'}\check{f}_{n'j}y_{n'}$ . It is not zero as the left and right factors in the product are non-zero by Corollary 4.3, and  $f_{nn'} = [2]_q f_n \neq 0$ . Note that restriction  $i \leq j'$  can be relaxed for symplectic  $\mathfrak{g}$ .

The case of orthogonal  $\mathfrak{g}$  and  $j < s'$  is covered by Corollary 4.3, as  $\delta_j^- = 1$  then.

Now suppose that  $\mathfrak{g}$  is orthogonal and  $s' \leq j$  (implying  $i \leq s$ ). Projection modulo  $J_s$  yields  $\hat{f}_{ij}/\delta_j^- = \hat{f}_{in}(\hat{f}_{ss'}/\delta_j^-)\hat{f}_{n'j}$ . It is not zero provided  $\hat{f}_{ss'}/\delta_j^- \neq 0$ . We find it equal to  $f_n^2$  in for odd  $N$  and  $f_{n-1}f_n$  for even  $N$ . This completes the proof.  $\square$

This theorem gives regularization of the negative Mickelsson generators,  $\hat{z}_{-\alpha_j} = \hat{f}_{1j}$ , for  $\alpha = \varepsilon_1 - \varepsilon_j \in R^+$ .

**Example 4.5.** Let us illustrate Theorem 4.4 on the example of  $\check{f}_{15}$  for  $\mathfrak{g} = \mathfrak{so}(6)$ . The algebra  $\mathcal{A}_5$  is generated by  $y_1, \dots, y_4$  subject to  $y_3y_4 = y_2$ . Then  $d_5^- = \bar{A}_2 = \frac{y_2-1}{q-\bar{q}}$ . With  $f_{34} = 0$ ,  $\hat{f}_{15}$  reads

$$\hat{f}_{15} = f_{15}A_1 + f_{13}f_{35}A_{1,3} + f_{14}f_{45}A_{1,4} + f_{12}f_{25}A_{1,2} + f_{12}f_{23}f_{35}A_{1,2,3} + f_{12}f_{24}f_{45}A_{1,2,4}.$$

The elements  $f_1f_2f_3$ ,  $f_{13}f_3$ ,  $f_{14}f_2$  and  $f_{15}$  form a basis in the subspace of weight  $\varepsilon_5 - \varepsilon_1$  in  $U_q(\mathfrak{g}_-)$ . With  $f_{12} = f_1$ ,  $f_{23} = f_{45} = f_2$ ,  $f_{24} = f_{35} = f_3$ , we rewrite  $\check{f}_{15} = \hat{f}_{15}\bar{A}_1 \dots \bar{A}_4$  as

$$\check{f}_{15} = f_{15}\bar{A}_2\bar{A}_3\bar{A}_4 + f_{13}f_3\bar{A}_4\bar{A}_2 + f_{14}f_2\bar{A}_3\bar{A}_2 + f_1f_2f_3(q - \bar{q} + A_3 + A_4)\bar{A}_3\bar{A}_4.$$

Taking in to account that  $(q - \bar{q} + A_3 + A_4)\bar{A}_3\bar{A}_4 = \frac{y_3y_4-1}{q-\bar{q}} = \bar{A}_2$ , we get the regularization

$$\check{f}_{15}/\delta_5^- = f_{15}\frac{y_3-1}{q-\bar{q}}\frac{y_4-1}{q-\bar{q}} + f_{13}f_3\frac{y_4-1}{q-\bar{q}} + f_{14}f_2\frac{y_3-1}{q-\bar{q}} + f_1f_2f_3,$$

which never turns zero.

## 5 Regularization of positive Mickelsson generators

In this section, we regularize Mickelsson generators of positive weights. The assignment  $f_\alpha \leftrightarrow e_\alpha$ ,  $q^{\pm h_\alpha} \mapsto q^{\mp h_\alpha}$  extends to an algebra automorphism  $\omega: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ . Denote  $e_{ji} = \omega(f_{ij}) \in U_q(\mathfrak{g}_+)$ .

Fix  $j > 1$ . Given  $j \prec l \leq N$ , define  $D_l^j = \frac{q^{\eta_{l1}-\eta_{j1}}}{[\eta_{j1}-\eta_{l1}]_q} \in \hat{U}_q(\mathfrak{h})$ . Set  $D_{\vec{m}}^j = D_{m_1}^j \dots D_{m_k}^j$  for a route  $\vec{m} = (m_1, \dots, m_k)$  with  $j \prec m$ . Define

$$\hat{z}_{\alpha_j} = e_{j1} + \sum_{j \prec \vec{m} \prec k \leq N}^{\emptyset} f_{j,\vec{m},k} e_{k1} D_{\vec{m},k}^j (-1)^{|j-k|} q^{\eta_{j1}-\eta_{k1}} \in \hat{U}_q(\mathfrak{g}), \quad j = 2, \dots, N.$$

and  $\check{z}_{\alpha_j} = \hat{z}_{\alpha_j} \prod_{j \prec l} \bar{D}_l^j \in U_q(\mathfrak{g})$ . The elements  $\hat{z}_{\alpha_j}$  with  $\alpha_j = \varepsilon_1 - \varepsilon_j \in R^+$  form a set of positive Mickelsson generators, [10].

Introduce an anti-algebra automorphism  $\tau$  of  $U_q(\mathfrak{g}_-)$  that is identical on the Chevalley generators. The Serre relations imply that  $\tau$  is well defined.

**Lemma 5.1.** *For all  $i, j$ ,  $\tau(f_{ij}) = f_{j'i'}$ .*

*Proof.* The proof can be conducted by considering explicit expressions (2.3-2.8) for  $f_{ij}$ . It is immediate for general linear and odd orthogonal  $\mathfrak{g}$ . In other cases, the hardest part of the proof boils down to checking  $[f_{jn}, f_{ni}]_{\bar{q}^{1+\delta_{ij'}}} = [f_{jn'}, f_{n'i}]_{\bar{q}^{1+\delta_{ij'}}$  for  $i < n$ ,  $n' < j$ ,  $\mathfrak{g} = \mathfrak{sp}(2n)$  and  $\mathfrak{g} = \mathfrak{so}(2n)$ . This can be done by applying the modified Jacobi identity

$$[x, [y, z]_a]_b = [[x, y]_c, z]_{\frac{ab}{c}} + c[y, [x, z]_{\frac{b}{c}}]_{\frac{a}{c}},$$

for appropriate values of  $a, b, c \in \mathbb{C}$ . They are chosen so as to kill one of the internal commutators, making use of the relations in the subalgebras of A-type in  $U_q(\mathfrak{g})$ .  $\square$

Introduce  $\delta_j^- \in \mathcal{A}_j$  as follows. Put  $\delta_j^- = 1$  for  $j < s'$  and, for  $j \geq s'$ ,  $\delta_j^- = \frac{y_{j'}-1}{q-\bar{q}} = \bar{A}_{j'}$  if  $\mathfrak{so}(2n)$  and  $\delta_j^- = \frac{y_{j'}^{\frac{1}{2}}+1}{q+1}$  if  $\mathfrak{so}(2n+1)$ .

Denote  $\delta_j^+ = \frac{q^{2\eta_{j1}-2\eta_{j'1}-1}}{q-\bar{q}}$  for  $\mathfrak{so}(2n)$ , and  $\delta_j^+ = \frac{qq^{2\eta_{j1}-2\eta_{*1}+1}}{q+1}$  for  $\mathfrak{so}(2n+1)$  assuming  $j \leq s$ . In all other cases including symplectic  $\mathfrak{g}$ , put  $\delta_j^+ = 1$ .

**Theorem 5.2.** *For each  $j = 2, \dots, N$  the element  $\check{z}_{\alpha_j}$  is divisible by  $\delta_j^+$ . The quotient  $\check{z}_{\alpha_j}(\lambda)/\delta_j^+(\lambda)$  does not turn zero at all  $\lambda \in \mathfrak{h}^*$ .*

*Proof.* We can assume that  $\mathfrak{g}$  is orthogonal and  $s' \leq j'$ . For all  $\lambda \in \mathfrak{h}^*$ , define  $\check{z}_{jk} \in U_q(\mathfrak{g}_-)$ ,  $k = j, \dots, N$ , setting  $\check{z}_{jj} = \prod_{j \prec l} \bar{D}_l^j$  and  $\check{z}_{jk} = \sum_{j \prec \bar{m} \prec k}^{\emptyset} f_{j, \bar{m}, k} D_{\bar{m}, k}^j \check{z}_{jj}$  for  $j \prec k$ . Then  $\check{z}_{\alpha_j}(\lambda) = \sum_{k=j}^N (-1)^{|j-k|} q^{\eta_{j1}-\eta_{k1}} \check{z}_{jk}(\lambda) e_{k1}$ . By the PBW theorem for  $U_q(\mathfrak{g})$ ,  $\check{z}_{\alpha_j}(\lambda) = 0$  if and only if  $\check{z}_{jk}(\lambda) = 0$  for all  $k$ .

The assignment  $y_{k'} = q^{2\eta_{j1}-\eta_{k1}}$ ,  $j \prec k$ , sends  $\delta_{j'}^-$  over to  $\delta_j^+$ . One can check that the algebra generated by  $\{y_l\}_{l \prec j'}$  is isomorphic to  $\mathcal{A}_{j'}$ . Indeed,  $y_k y_{k'} = y_j$  for all  $k \neq *$  and  $q^2 y_*^2 = y_j$  (odd  $N$ ).

The anti-automorphism  $\tau$  takes  $\check{z}_{jk}$  to  $\check{f}_{k'j'}(\prod_l \bar{A}_l)$ , where the product is done over those  $l < j'$  which  $k' \not\leq l$ ; it includes all  $\bar{A}_l$  with  $k' < l$ . By Lemma 3.5,  $\check{f}_{k'j'}$  is divisible by  $\delta_{j'}^-$  if  $k' \leq j$ , otherwise  $\prod_l \bar{A}_l$  contains  $\bar{A}_j$  divisible by  $\delta_{j'}^-$ . This proves that  $\check{z}_{\alpha_j}$  is divisible by  $\delta_j^+$ . Finally, the element  $\check{f}_{1j'}/\delta_{j'}^-$  is the  $\tau$ -image of  $\check{z}_{jN}/\delta_j^+$ , up to the factor  $(-1)^{|j-k|} y_{k'} \neq 0$ . Since  $\check{f}_{1j'}/\delta_{j'}^-$  does not vanish at all weights,  $\check{z}_{\alpha_j}/\delta_j^+ \neq 0$  at all weights too.  $\square$

## 6 Application: decomposition of $V \otimes M_\lambda$

Define an ascending sequence of submodules  $W_j = \sum_{i=1}^j M_i \subset V \otimes M_\lambda$ , for all  $j = 1, \dots, N$ . Lemma 3.3 implies that  $W_j \subset V_j$ . We apply the regularization analysis to answer the question when  $V \otimes M_\lambda$  is a direct sum of  $M_j$ . Clearly that is the case if the eigenvalues  $x_j$  of  $\mathcal{Q}$  are pairwise distinct. However the converse is not true.

**Proposition 6.1.** *For all  $j \in [1, N]$ , the following statements are equivalent: i)  $V_j = W_j$ , ii)  $V_i = W_i$  for all  $i \leq j$ , iii) projection  $\wp_i: M_i \rightarrow V_i/V_{i-1}$  is an isomorphism for all  $i \leq j$ , iv)  $W_j = \oplus_{i=1}^j M_i$ .*

*Proof.* Since all  $M_i$  and  $V_i/V_{i-1}$  are Verma modules of the same highest weight, iii) is equivalent to  $\wp_i$  being surjective or injective. The implication ii)  $\Rightarrow$  i) is trivial. With  $W_1 = V_1$ , assume that ii) is violated and let  $k > 1$  be the smallest such that  $W_k \neq V_k$ . Then  $V_k$  and  $W_k$  have different multiplicities of weight  $\lambda + \varepsilon_k$ , so that i)  $\Rightarrow$  ii).

Assuming ii) we find that all maps  $\wp_i$  are surjective, hence iii). Conversely, iii) implies that all maps  $W_i \rightarrow V_i/V_{i-1}$  are surjective. Since,  $W_1 = V_1$ , induction on  $i$  proves ii).

If  $\wp_i$  are injective, then  $M_i \cap W_{i-1} \subset M_i \cap V_{i-1} = \{0\}$ , that is, iii)  $\Rightarrow$  iv). Finally, suppose iv) and prove iii) by induction: assuming iii)  $\Leftrightarrow$  ii) true for  $i < j$ , the map  $\wp_j$  is injective and hence an isomorphism.  $\square$

Let  $u_j = \tilde{F}_j/d_j^- \in V \otimes M_\lambda$  be the regularized singular vector and introduce  $C_j(\lambda) \in \mathbb{C}$  through the equality  $\wp_j(u_j) = C_j(\lambda)v_{\lambda,j}$ .

**Corollary 6.2.** *The module  $V \otimes M_\lambda$  is a direct sum  $\oplus_{j=1}^N M_j$  if and only if all  $C_j(\lambda) \neq 0$ .*

*Proof.* Since  $u_j$  does not vanish at all  $\lambda$ ,  $C_j(\lambda) = 0$  if and only if  $\wp_j$  is an isomorphism.  $\square$

Denote  $\phi_{ij} = \frac{q^{2\xi_{ij}} - 1}{q - \bar{q}}$  for  $i < j$  apart from  $i = j'$  for  $\mathfrak{g} = \mathfrak{so}(N)$ , in which case we set  $\phi_{j'j} = 1$  for even  $N$  and  $\varphi_{j'j} = \frac{q^{\xi_{j'j}} - 1}{q - \bar{q}}$  for odd  $N$ . Then  $C_j \simeq \prod_{i=1}^{j-1} \phi_{ij}$  up to a numerical factor that never turns zero. Recall that  $q^{2\xi_{ij}} = x_i x_j^{-1}$ , where  $x_i$  are the eigenvalues of  $\mathcal{Q}$ . If they are all distinct, then clearly  $V \otimes M_\lambda = \oplus_{j=1}^N M_j$ . It follows that the converse is also true for symplectic  $\mathfrak{g}$ . For orthogonal  $\mathfrak{g}$ , the spectrum of  $\mathcal{Q}$  does not separate the submodules  $M_j$  at all weights. For instance, choose a weight  $\lambda$  such that  $q^{2(\lambda + \rho, \varepsilon_j)} = -1$ . Then  $x_j x_{j'}^{-1} = q^{4(\lambda + \rho, \varepsilon_j)} = 1$ . If all other  $x_l$  are pairwise distinct and different from  $x_j$ , then the direct sum decomposition still holds. This phenomenon facilitated quantization of borderline Levi conjugacy classes in [10].

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